Lecture 13
14.4 The Chain Rule

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Last class

Definition

The partial derivative of $f(x, y)$ with respect to x is

$$f_x(x, y) = \frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}.$$

The partial derivative with respect to y is

$$f_y(x, y) = \frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}.$$

$$\frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\right] = \frac{\partial^2 f}{\partial x^2} = f_{xx} \qquad \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right] = \frac{\partial^2 f}{\partial y \partial x} = f_{xy}$$

$$\frac{\partial}{\partial y}\left[\frac{\partial f}{\partial y}\right] = \frac{\partial^2 f}{\partial y^2} = f_{yy} \qquad \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right] = \frac{\partial^2 f}{\partial x \partial y} = f_{yx}$$

Example

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Find the second-order partial derivatives of $f(x, y) = ye^{x^2 - y}$.

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$f_x(x,y) = 2xye^{x^2-y}$
$f_y(x,y) = e^{x^2-y} - ye^{x^2-y}$
$f_{xx}(x,y) = (4x^2+2)e^{x^2-y}$
$f_{xy}(x,y) = (-2x)(y-1)e^{x^2-y}$
$f_{yx}(x,y) = (-2x)(y-1)e^{x^2-y}$
$f_{yy}(x,y) = (y-2)e^{x^2-y}$

Mixed partials theorem

Theorem

If $f(x, y)$ and its partial derivatives f_x, f_y, f_{xy}, f_{yx} are defined near (a, b), then

$$f_{yx}(a, b) = f_{xy}(a, b).$$

This is known as Clairaut's Theorem.

In particular, if the conditions in the theorem hold for all the pairs (a, b) in the domain of the functions involved, then the functions will have the same formula on that domain, and we only need to find one of f_{xy} or f_{yx} to know the other.

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For multi-variable functions, the same rules apply, but we also have to add across the various variables. There are several cases to consider.

One independent variable

Theorem

If $w = f(x, y)$ is differentiable and if $x = x(t)$, $y = y(t)$ are differentiable functions of t, then the composite $w = f(x(t), y(t))$ is a differentiable function of t and

$$\frac{dw}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}.$$

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We have $w = f(x(t), y(t), z(t))$ is a function of three intermediate variables and one independent variable. Using a generalization of the formula above, we calculate

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial w}{\partial y}\cdot\frac{dy}{dt}+\frac{\partial w}{\partial z}\cdot\frac{dz}{dt} = (y)(-\sin(t))+(x)(\cos(t))+1(1)$$

$$= (\sin(t))(-\sin(t))+(\cos(t))(\cos(t))+1 = -\sin^2(t)+\cos^2(t)+1.$$

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$$= (\sin(t))(-\sin(t))+(\cos(t))(\cos(t))+1 = -\sin^2(t)+\cos^2(t)+1.$$

This is the same formula that would be obtained if we had substituted x, y, z into w first to get $w = \cos(t)\sin(t) + t$ and then differentiated.

Two independent variables

Theorem

If $w = f(x, y)$ is differentiable and if $x = x(s, t)$, $y = y(s, t)$ are differentiable functions of s and t, then the composite $w = f(x(s, t), y(s, t))$ is a differentiable function of s and t and

$$\frac{\partial w}{\partial t} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t}.$$

$$\frac{\partial w}{\partial s} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s}.$$

Example

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Find $\dfrac{\partial w}{\partial r}$ *and* $\dfrac{\partial w}{\partial s}$ *in terms of* r *and* s *if* $w = x + 2y + z^2$, $x = \frac{r}{s}$, $y = r^2 + \ln(s)$, $z = 2r$

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We have

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r} = (1)(\frac{1}{s}) + (2)(2r) + (2z)(2) = \frac{1}{s} + 12r$$

and

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$

$$= (1)(-\frac{r}{s^2}) + (2)(\frac{1}{s}) + (2z)(0) = \frac{-r + 2s}{s^2}$$